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Investigation of a class of stochastic processes using path-integral techniques

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Abstract. A study is made of the Langevin equation $\dot{x} = -V'(x) + g(x)\xi(t) + \eta(t)$, where the noises $\eta(t)$ and $\xi(t)$ are Gaussian with zero mean and with $\langle\eta(t)\eta(t')\rangle = 2R\delta(t-t')$, $\langle\xi(t)\xi(t')\rangle = (D/\tau)\exp -|t-t'|/\tau$. Path integral representations for conditional probability distributions are given for the two cases $\tau = 0$ and $\tau \neq 0$. For R and D small, but of the same order, the appropriate path integrals are evaluted to leading order using the method of steepest descents, in order to find the stationary probability distribution $P_{st}(x)$ and the mean relaxation time $\bar{T}$ for escape from a potential well. Analytical expressions are given for $\bar{T}$ when $\tau = 0$ and when τ is small. For general τ we present numerical results for the stationary probability distribution, using the particular forms $V(x) = -\frac{1}{2}ax^2 + \frac{1}{4}Ax^4 - R\ln x$ and $g(x) = x$, which are appropriate to the dye laser.

1. Introduction

The effect that non-white (i.e. coloured) external noise can have on physical systems has been of considerable interest in recent years. The motivation has been both physical and mathematical; in the latter case the challenge presented by the non-Markovian nature of the problem being the main driving force [1]. However, the difficulties inherent in generalizing the familiar tools of the theory of stochastic processes to this situation has meant that the majority of authors have restricted their attention to the simplest model: a Langevin equation with additive coloured noise. On the other hand, the use of path-integral techniques for calculating probability distributions in the weak noise limit [2] can be used irrespective of whether the noise is coloured or not, and seems to be the most efficient way of performing calculations on more complex model systems [3, 4].

In this paper we use the path integral approach to study the model defined by the Langevin equation

$$\dot{x} = -V'(x) + g(x)\xi(t) + \eta(t). \tag{1}$$

Here $\xi(t)$ and $\eta(t)$ are two different types of noise; the former is coloured while the latter is white. We will assume that they are both Gaussian with zero mean. There are several reasons why models of the type (1) are of interest. Firstly, coloured noise is frequently multiplicative, rather than additive, and it seems desirable to investigate the effects of having a non-trivial function $g(x)$ multiplying $\xi(t)$. Secondly, internal noise will typically also be present and so it seems natural to include a white noise $\eta(t)$ in addition to the coloured noise $\xi(t)$. In fact, as we will see later, the process with the white noise absent is a singular one in the path integral formulation. Thirdly,

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there are definite physical problems which can be modelled by (1), that is, situations which require both multiplicative coloured noise and additive white noise. We mention here noise in dye lasers [5], where $x(t)$ represents the magnitude of the complex laser field [6, 7], and the problem of escape over a fluctuating barrier that occurs in highly constrained systems such as a glass [8].

Since both $\xi(t)$ and $\eta(t)$ are Gaussian with zero mean, they are completely specified by their second moments. These will be defined by

$$\langle\xi(t)\xi(t')\rangle=\frac{D}{\tau}\exp(-|t-t'|/\tau) \tag{2}$$

and

$$\langle\eta(t)\eta(t')\rangle=2R\delta(t-t'). \tag{3}$$

We have taken the noise to be exponentially correlated, since this represents the simplest departure from the white noise. However, it is not difficult to extend the treatment to more general types of correlator [3]. The quantities we wish to calculate, such as the rate of escape of a particle over a potential barrier or various probability distributions, are found by evaluating particular path integral representations in the limit of small diffusion constants by the method of steepest descents. We will only be interested in the leading exponential contributions to these quantities in this paper. The precise form this calculation takes depends on the relative sizes of the two diffusion constants D and R; we will frequently assume that they are of the same order for illustrative purposes. Finally, although we wish to keep our discussion as general as possible, we will eventually have to assume some specific form for $V(x)$ and $g(x)$ when carrying out numerical computations. In this case we will take the forms appropriate in the study of the dye laser, that is $V(x)=-\frac{1}{2}ax^2+\frac{1}{4}Ax^4-R\ln x$ and $g(x)=x$.

The outline of the paper is as follows. In section 2 we obtain a path integral representation for the conditional probability density function $P(x, \xi, t\,|\,x_0, \xi_0, t_0)$ for $\tau\neq 0$ as well as a simpler representation for $\tau=0$. In section 3 we calculate the mean escape time and the stationary probability distribution for $\tau=0$ and in section 4 we develop a power series expansion in τ about the $\tau=0$ result for the mean escape time. The model for general τ is dealt with in section 5 where we present numerical results for the dye laser.

2. Path integral formulation

In this section we develop a path integral description in terms of the stochastic process defined by (1)-(3) by first introducing an equivalent two-dimensional Markov process and then writing down the corresponding Fokker-Planck equation. The two-dimensional process is given by

$$\dot{x}=-V'(x)+g(x)\xi+\eta_1(t) \qquad \dot{\xi}=-\tau^{-1}\xi+\tau^{-1}\eta_2(t) \tag{4}$$

where $x(t)$ and $\xi(t)$ are initially uncorrelated and where the $\eta_i(t)$ are Gaussian with zero mean and with

$$\langle\eta_i(t)\eta_j(t')\rangle=2D_i\delta_{ij}\delta(t-t') \qquad i,j=1,2. \tag{5}$$

In the notation of section 1, $R=D_1$ and $D=D_2$. To see that the above process is equivalent to the one defined in section 1, we first solve the second equation in (4) with the initial condition on $\xi(t)$ set in the infinitely distant past. We find that $\langle\xi(t)\rangle=0$ and $\langle\xi(t)\xi(t')\rangle$ has the form (2). Furthermore, since $\eta_2(t)$ is Gaussian, so is $\xi(t)$. Hence the two processes are equivalent. If, instead, $x(t)$ and $\xi(t)$ are specified at some initial time t_0 then the result holds provided these two quantities were uncorrelated initially, since their previous history will then be irrelevant [9].

Since the process (4) is Markovian we may describe it using a Fokker–Planck equation. To simplify notation, we introduce the two-component vector $z=(z_1, z_2)=(x, \xi)$, and write (4) as

$$\dot{z}_1 = A_1(z)+\eta_1(t) \qquad \dot{z}_2 = A_2(z)+\tau^{-1}\eta_2(t) \tag{6}$$

where the $A_i(z)$ are given by

$$A_1(z) = -V'(z_1)+z_2 g(z_1) \qquad A_2(z) = -\tau^{-1}z_2. \tag{7}$$

The equation satisfied by the conditional probability density function $P(z, t\,|\,z_0, t_0)$ is then [10]

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{2}\frac{\partial}{\partial z_i}[A_i(z)P]+\tfrac{1}{2}\sum_{i,j=1}^{2}\frac{\partial^2}{\partial z_i \partial z_j}[B_{ij}P] \tag{8}$$

where B is the (constant) two by two matrix

$$B=\begin{pmatrix} 2D_1 & 0 \\ 0 & 2D_2\tau^{-2}\end{pmatrix}. \tag{9}$$

It is now straightforward to write the solution of (8) in terms of a path integral. Since B is a constant, that is, the noise is not multiplicative, ambiguities associated with time-discretization are at a minimum, and one finds [11]

$$P(z, t\,|\,z_0, t_0) = \int_{z(t_0)=z_0}^{z(t)=z} \mathscr{D}z \exp(-S[z]/D)J[z] \tag{10}$$

where

$$S[z]=\frac{D}{4D_1}\int_{t_0}^{t} \mathrm{d}t(\dot{z}_1+V'(z_1)-z_2 g(z_1))^2+\frac{D\tau^2}{4D_2}\int_{t_0}^{t}\mathrm{d}t(\dot{z}_2+\tau^{-1}z_2)^2 \tag{11}$$

$$J[z]=\exp\left(\tfrac{1}{2}\int_{t_0}^{t}\mathrm{d}t(V''(z_1)+\tau^{-1}-g'(z_1)z_2)\right) \tag{12}$$

and

$$\mathscr{D}z=\lim_{\varepsilon\to 0}\prod_{i=1}^{N-1}\left\{\frac{\tau\,\mathrm{d}z_i}{4\pi\varepsilon(D_1D_2)^{1/2}}\right\}\left[\frac{\tau}{4\pi\varepsilon(D_1D_2)^{1/2}}\right]. \tag{13}$$

A few comments are in order. Firstly, D has been introduced simply to indicate the relative importance of the various terms in the small noise limit: if $D_1 \sim D_2$ $(\sim D)$ are small, then the dominant part of the integrand is that involving the 'action' $S[z]$. Secondly, $J[z]$ is, up to normalization, the Jacobian factor which comes from the functional change of variable from $\{\eta_1(t), \eta_2(t)\}$ to $\{z(t)\}$. A definite choice of time discretization has been made, which in our case means that the coefficient multiplying the integral in the exponential (12) is $\frac{1}{2}$. Thirdly, this time discretization into N intervals of duration ε is evident in the definition of the measure (13). Here the subscript i labels the intervals and the limit consists of taking $\varepsilon\to 0$, $N\to\infty$ but with $N\varepsilon=(t-t_0)$

fixed. Finally, notice that if the white noise had been absent in (1), then the Fokker–Planck equation (8) would have contained a matrix B given by (9) but with $D_1 = 0$. This matrix would not have been invertible and the construction leading to (11)–(13) would not have been immediately applicable. It is in this sense that the process (1) without white noise is singular; a different procedure for obtaining a path integral representation for the solution of the Fokker–Planck equation has then to be adopted [9].

In later sections we will discuss the calculation of (10) in the small noise limit by the method of steepest descents, for general τ. However, as an introduction we begin by studying the white noise limit. The action $S[z]$ has a finite limit as $\tau \to 0$ which, moreover, is Gaussian in z_2. Hence the variable z_2 can be integrated out, leaving the reduced action

$$S[z_1] = \frac{D}{4}\int_{t_0}^{t} \mathrm{d}t \frac{(\dot{z}_1 + V'(z_1))^2}{(D_1 + D_2 g^2(z_1))}. \tag{14}$$

If we wish to determine the form of the integral beyond the leading order term (14), we have to include the $g'(z_1)z_2$ factor that occurs in $J[z]$ in the integration and also cope with the non-trivial $\tau \to 0$ limit in (12) and (13). Rather than following this route, it is easier to go back to (1), where now both $\xi(t)$ and $\eta(t)$ are white, and to write down a Fokker–Planck equation which is equivalent to this Langevin equation. One finds that $P(x, t|x_0, t_0)$ satisfies the equation [10]

$$\begin{aligned}\frac{\partial P}{\partial t} &= -\frac{\partial}{\partial x}[A(x)P] + \frac{\partial}{\partial x}\left[\phi(x)\frac{\partial}{\partial x}\phi(x)P\right] \\ &= -\frac{\partial}{\partial x}[(A(x) + \phi(x)\phi'(x))P] + \frac{\partial^2}{\partial x^2}[\phi^2(x)P]\end{aligned} \tag{15}$$

where

$$A(x) = -V'(x) \tag{16}$$

and

$$\phi(x) = \sqrt{D_1 + D_2 g^2(x)}. \tag{17}$$

This process is thus equivalent to one described by the equation

$$\dot{x} = A(x) + \phi(x)\zeta(t) \tag{18}$$

where $\zeta(t)$ is a Gaussian white noise with zero mean and unit diffusion constant. The structure of $\phi(x)$ is exactly what we would expect to get when we add two independent Gaussian processes together. We have used the Stratonovich prescription in defining (1) or equivalently (18). Since the noise is multiplicative and white, it is crucial to give an appropriate interpretation rule in addition to the Langevin equation [12]. We naturally use the Stratonovich prescription since we are interested here in the $\tau \to 0$ limit of a non-white process.

The solution to the Fokker–Planck equation (15) may be written as [11]

$$P(x, t|x_0, t_0) = \int_{x(t_0)=x_0}^{x(t)=x} \mathscr{D}x \exp(-S[x]/D) J[x] \tag{19}$$

where

$$S[x] = \frac{D}{4} \int_{t_0}^{t} \mathrm{d}t \frac{(\dot{x} + V'(x))^2}{\phi^2(x)} \tag{20}$$

$$J[x] = \exp\left(\tfrac{1}{2} \int_{t_0}^{t} \mathrm{d}t\, \phi(x) \frac{\mathrm{d}}{\mathrm{d}x} [V'(x)/\phi(x)] \right) \tag{21}$$

and

$$\mathscr{D}x = \lim_{\varepsilon \to 0} \prod_{i=1}^{N-1} \left\{ \frac{\mathrm{d}x_i}{(4\pi\varepsilon\phi^2(x_i))^{1/2}} \right\} (4\pi\varepsilon\phi^2(x))^{-1/2}. \tag{22}$$

Similar comments to those made following (13) are also applicable here. The leading order result (20) agrees with (14), but (21) and (22) show the greater complexity of the Jacobian and measure terms when the noise is multiplicative.

3. Calculation in the white noise limit

The path integral representation for the conditional probability for $\tau = 0$ derived in the previous section is used in this section to calculate the escape time and stationary probability distribution. First, suppose $V(x)$ is a double-well potential. We can ask what is the mean time taken for a noise-induced transition from one well to the other to occur. We expect this to be exponentially large in the diffusion constants, if they are small and of the same order. Of course, since the process is one-dimensional and Markovian, we could easily calculate this quantity directly from the Fokker–Planck equation using standard methods [10]. However, our intention is to perform the calculation using a path-integral approach which can be applied, without any great modification, to the $\tau \neq 0$ case, so we wish to deduce this mean transition time from (19). Suppose, then, that $D_1 = \sigma_1 D$ and $D_2 = \sigma_2 D$ where σ_1 and σ_2 are of order unity. Then the path integral (19) may be evaluated by the method of steepest descent to give

$$P(x, t|x_0, t_0) \sim \exp(-S[x_c]/D) \tag{23}$$

where $x_c(t)$ is the extremal path connecting the two wells. In this paper we will only concern ourselves with evaluating quantities to leading order; the next order calculation which gives the prefactor in (23) is considerably more difficult. The extraction of the mean transition time from (23) has been discussed elsewhere [4]; however, the basic points are relatively simple and can be quickly summarized. A more complete calculation of $P(x, t|x_0, t_0)$ would yield a power series in $T\Phi \exp(-S[x_c]/D)$, where $T = t - t_0$ and Φ is a prefactor independent of T and D [13]. The quantity displayed on the right hand side (RHS) of (23) is merely the first term in this series and comes from a simple extremal path between the two wells. The other terms in the power series come from multiple paths between the wells. The important point is that $P(x, t|x_0, t_0)$ is simply a function of $T/\bar{T}$ where $\bar{T} = \Phi^{-1} \exp(S[x_c]/D)$. Since $\bar{T}$ is the required mean time (up to constants of order unity which do not concern us here) we can obtain it to leading order simply by taking the inverse of the RHS of (23).

We are therefore left with performing the variation

$$0 = \delta S = \delta \left[\tfrac{1}{4} \int_{t_0}^{t} \mathrm{d}t \frac{(\dot{x} + V'(x))^2}{(\sigma_1 + \sigma_2 g^2(x))} \right]. \tag{24}$$

Since the white noise limit of (1) is equivalent to the single white noise process (18), this problem has already been studied in [4]. There, the external paths were shown to be $\dot{x} = \pm V'(x)$, the positive sign being the appropriate one for 'uphill' paths from the bottom of a well to the top of the barrier and the negative sign the for the 'downhill' paths from the top of the barrier to the bottom of a well. The latter path gives zero contribution to the action $S[x_c]$ as we would expect and so

$$S[x_c] = \int_{x_0}^{x_1} dx \frac{V'(x)}{(\sigma_1 + \sigma_2 g^2(x))} \tag{25}$$

where x_0 is the x-coordinate of the bottom of the starting well and x_1 is the x-coordinate of the top of the barrier. Therefore

$$\begin{aligned} \bar{T} &\sim \exp\left(\frac{1}{D}\int_{x_0}^{x_1} dx \frac{V'(x)}{(\sigma_1 + \sigma_2 g^2(x))}\right) \\ &= \exp\left(\int_{x_0}^{x_1} dx \frac{V'(x)}{(D_1 + D_2 g^2(x))}\right). \end{aligned} \tag{26}$$

Another quantity we may calculate by applying the method of steepest descents to (19) is the stationary probability distribution

$$P_{st}(x) = \lim_{t_0 \to -\infty} P(x, t | x_0, t_0) \tag{27}$$

where x_0 is again a local minimum of the potential. The result is of the form (23). The extremal path once more satisfies $\dot{x} = \pm V'(x)$; all that changes is the boundary conditions. These are $x(-\infty) = x_0$ and $x(0) = x$, where we have taken the final time to be $t = 0$, without loss of generality. From (23), (25) and (27) we see that

$$\begin{aligned} P_{st}(x) &\sim \exp(-S(x)/D) \\ &= \exp\left(-\int_{x_0}^{x} dx \frac{V'(x)}{(D_1 + D_2 g^2(x))}\right). \end{aligned} \tag{28}$$

Once again, while (28) is easily obtained as a time-independent solution of the Fokker–Planck equation (15), the method adopted here only shows its true potential in the case $\tau \neq 0$ as we will show in the following sections.

Finally, let us illustrate these ideas on the dye laser problem where $V(x) = -\frac{1}{2}ax^2 + \frac{1}{4}Ax^4 - R \ln x$ and $g(x) = x$. To leading order the $-R \ln x$ term is omitted. Also since $x(t) = |E(t)|$, where $E(t)$ is the complex laser field, $x \geqslant 0$, and so for $a > 0$, $V(x)$ has only one potential well. This means that only the result for the stationary probability distribution is relevant in this case and we have simply to evaluate the integral

$$\int_{x_0}^{x} dx \frac{(-ax + Ax^3)}{(R + Dx^2)} \tag{29}$$

where $x_0 = (a/A)^{1/2}$. This leads to

$$P_{st}(x) \sim \exp\left(-\frac{Ax^2}{2D} + \frac{\lambda}{2D} \ln(x^2 + RD^{-1})\right) \tag{30}$$

where $\lambda = a + A(RD^{-1})$, in agreement with [6].

4. The small τ expansion

In this section we develop a power series expansion in τ about the $\tau = 0$ limit in order to extend the white noise result for the mean escape time (26) to small τ.

The starting point for $\tau \neq 0$ is the expression for the conditional probability distribution $P(z, t|z_0, t_0)$ given in (10) in terms of the action $S[z]$ given by (11). In what follows we will return to using the variables (x, ξ) rather than (z_1, z_2); it will also be convenient to introduce the velocity $y(x) = \dot{x}(t)$. Since the action $S[z]$ contains no explicit time dependence we can change from the independent variable t to x so that $\dot{\xi}(t) = y(x)\xi'(x)$ and

$$S[y, \xi] = \frac{1}{4\sigma_1}\int_{x_0}^{x} \frac{\mathrm{d}x}{y}(y + V' - \xi g)^2 + \frac{1}{4\sigma_2}\int_{x_0}^{x} \frac{\mathrm{d}x}{y}(\xi + \tau y \xi')^2 \tag{31}$$

where primes represent differentiation with respect to x and $x_0 = x(t_0)$. As in the previous section we will assume that σ_1 and σ_2 are both of order unity. Also, we will take x_0 to be the x-coordinate of some local minimum of the potential $V(x)$ since this is relevant to the calculation of both escape times and stationary probability distributions, hence $V'(x_0) = 0$.

As in the white noise case the calculation of quantities in the small D limit depends on finding the extremal path between two points; in the case of the escape time these two points are the bottom and top of a barrier while for the stationary probability distribution at x they are a local minimum of the potential and the point x itself. The extremal path x_c is the one that minimizes the action (31). Since the action depends on both $y(x)$ and $\xi(x)$ the minimal action is found by varying with respect to these two variables. Setting the variations equal to zero results in the following two coupled differential equations:

$$\xi^2 - \tau^2 y^2 \xi'^2 = \frac{\sigma_2}{\sigma_1}(y^2 - (V' - g\xi)^2) \tag{32}$$

and

$$\xi - \tau^2(y^2\xi'' + yy'\xi') = \frac{\sigma_2}{\sigma_1} g(y + V' - g\xi). \tag{33}$$

In principle it is possible to eliminate y from the above equations to obtain a single second-order equation in ξ. This tells us that to obtain a solution to the equations requires two independent boundary conditions reflecting the fact that we are dealing with a two-dimensional Markov process (4).

The white noise limit is taken by setting τ equal to zero in (32) and (33) and solving the resulting pair of simultaneous equations in y and ξ. This is easily done and one finds two sets of solutions:

$$y = -V'(x) \qquad \xi = 0 \tag{34}$$

and

$$y = V'(x) \qquad \xi = \frac{2\sigma_2 g(x) V'(x)}{\sigma_1 + \sigma_2 g(x)^2}. \tag{35}$$

The latter solution is the one corresponding to the 'uphill' path and gives the action for the extremal path as

$$S(x)|_{\tau=0}=\int_{x_0}^{x}\mathrm{d}x\frac{V'(x)}{\sigma_1+\sigma_2 g^2} \tag{36}$$

which leads to the previously derived results (26) and (28) for the mean escape time to a neighbouring well and the stationary probability distribution.

To go beyond white noise to small τ it is natural to attempt an expansion about the white noise limit in powers of τ^2 since this coefficient enters into (32) and (33). We let

$$y=\sum_{n=0}^{\infty} y_n\tau^{2n} \tag{37}$$

and

$$\xi=\sum_{n=0}^{\infty} \xi_n\tau^{2n}. \tag{38}$$

The coefficients of the expansions are found by substituting these expressions into (32) and (33) and equating powers of τ^{2n} for $n=0, 1, 2, \ldots$. (Obviously, for uphill paths, y_0 and ξ_0 are given by (35).)

To leading order, the mean time of escape from one well to a neighbouring well is

$$\bar{T}\sim\exp(S[y(x_c), \xi(x_c)]/D) \tag{39}$$

where x_c is the extremal path from the bottom of the starting well (with x-coordinate x_0) to the top of the barrier (with x-coordinate x_1). The extremal path $x_c(t)$ which dominates the path integral for $P(z, t|z_0, t_0)$ is one for which $\dot{x}(t)$ and higher derivatives vanish in the limit $t_0\to-\infty$, $t\to\infty$. Since corrections to the action arising from the fact that $t-t_0$ is large, but finite, are exponentially small [2], [13], we take the path to be defined on the infinite time interval and so $y(x_0)=\dot{x}(-\infty)=0$ and $y(x_1)=\dot{x}(-\infty)=0$. It is obvious from (32) and (33) that this is equivalent to the boundary conditions $\xi(x_0)=0$ and $\xi(x_1)=0$. Due to these boundary conditions, the surface terms in the action (31) are zero and the action of the extremal path can be written as

$$S=\frac{1}{4\sigma_1}\int_{x_0}^{x_1}\frac{\mathrm{d}x}{y}(y+V'-\xi g)^2+\frac{1}{4\sigma_2}\int_{x_0}^{x_1}\frac{\mathrm{d}x}{y}(\xi^2+\tau^2y^2\xi'^2). \tag{40}$$

Substituting (37) and (38) into (40) gives

$$S=S|_{\tau=0}+\frac{\tau^2}{4\sigma_2}\int_{x_0}^{x_1}\mathrm{d}x\, V'(x)(\xi_0'(x))^2+\mathrm{O}(\tau^4) \tag{41}$$

where $S|_{\tau=0}$ is given by (25). It is worth pointing out that calculation of the action to $\mathrm{O}(\tau^{2n+2})$ requires the solution to $\mathrm{O}(\tau^{2n})$ of (32) and (33), hence the coefficient of τ^2 in (41) depends only on the zeroth order functions y_0 and ξ_0 in the expansions (37) and (38). We have obtained the action to $\mathrm{O}(\tau^4)$ but we do not write down the expression here as it is somewhat unwieldy. Instead, we now go on to study the special case of the dye laser for general τ.

5. Results for general τ

For general τ equations (32) and (33) must be solved numerically, for a specific choice of functions $V(x)$ and $g(x)$, since the solution for the 'uphill' extremal path cannot be obtained in closed form. We shall use the potential for the dye laser problem which, after an appropriate rescaling of x, t and V, can be expressed as

$$V(x) = -\tfrac{1}{2}x^2 + \tfrac{1}{4}x^4. \tag{42}$$

The term $-R \ln x$, which makes no contribution to leading order, has been omitted. We shall also take $g(x) = x$, the appropriate multiplicative factor in this case, and $\sigma_1 = \sigma_2$. The stationary probability distribution for the dye laser problem was derived in section 3 for $\tau = 0$ and is given in (30). Our aim in this section is to find the stationary probability distribution $P_{st}(x)$ numerically for general τ.

The conditional probability distribution $P(x, \xi, t|x_0, \xi_0, t_0)$ for the two variable process $\{x, \xi\} \equiv \{z\}$ is given in (10). The stationary probability distribution $P_{st}(x, \xi) = \lim_{t_0 \to -\infty} P(x, \xi, t|x_0, \xi_0, t_0)$ may be found by applying the method of steepest descents to (10) giving

$$P_{st}(x, \xi) \sim \exp(-S(x, \xi)/D) \tag{43}$$

where $S(x, \xi)$ is the action of the extremal path (y_c, ξ_c) linking the points (y_0, ξ_0) and (y, ξ) together with the boundary conditions $y(x_0) = 0$, or equivalently $\xi(x_0) = 0$ corresponding to $\dot{x}(-\infty) = 0$, and $\xi(x) = \xi$.

To find the marginal probability distribution $P_{st}(x)$ it is necessary to integrate out ξ from (43). Since we are working in the limit of small D it is natural to evaluate this integral by a second application of the method of steepest descents:

$$P_{st}(x) \sim \int_{-\infty}^{\infty} d\xi \exp(-S(x, \xi)/D) \sim \exp(-S(x, \xi_m)/D) \tag{44}$$

where ξ_m is the value of ξ which minimizes $S(x, \xi)$. Hence we have the form

$$P(x) \sim \exp(-S(x)/D) \tag{45}$$

with $S(x) = S(x, \xi_m)$. Since $S(x, \xi_m)$ is at a minimum with respect to ξ we must have that

$$\left.\frac{\partial S(x, \xi)}{\partial \xi}\right|_{\xi = \xi_m} = 0. \tag{46}$$

To find this partial derivative we consider the variation of the action $S[y, \xi]$ about the extremal path, keeping x_0 and x fixed but letting the value of ξ at the endpoint x vary. In general there are two contributions to this variation, one coming from the variation of the action with fixed boundaries, and the other from surface terms which arise when integrating by parts. In terms of the Lagrangian $L(y, \xi, \xi')$ defined by

$$S[y, \xi] = \int_{x_0}^{x} dx\, L(y, \xi, \xi') \tag{47}$$

the required variation is

$$\int_{x_0}^{x} dx\, \delta L = \int_{x_0}^{x} dx \left(\frac{\partial L}{\partial y} \delta y + \frac{\partial L}{\partial \xi} \delta \xi - \frac{d}{dx}\left(\frac{\partial L}{\partial \xi'}\right) \delta \xi \right) + \left.\frac{\partial L}{\partial \xi'} \delta \xi \right|_{x}. \tag{48}$$

The integral on the RHS of this expression is just the variation of the action at fixed boundaries, which is zero by definition of the extremal path. The second term makes the only contribution to the full variation and hence

$$\frac{\partial S(x, \xi)}{\partial \xi} = \left.\frac{\partial L(y, \xi, \xi')}{\partial \xi'}\right|_{x}. \tag{49}$$

The partial derivative on the RHS of (49) is taken using the integrand in (31) as the expression for the Lagrangian and the result combined with (46) gives

$$\xi_m + \tau y_m \xi'_m = 0 \tag{50}$$

where y_m is the value of y corresponding to $\xi = \xi_m$.

In principle we can now evaluate $S(x)$ by first solving (32) and (33) in the range (x_0, x) together with the boundary conditions

$$\xi(x_0) = 0 \tag{51}$$

$$\xi(x) + \tau y(x)\xi'(x) = 0 \tag{52}$$

and substituting the solution into (31).

However the equations (32) and (33) as they stand are in an unsuitable form for numerical analysis. To bring them into a more manageable form it is necessary to perform a change of variable from $y(x)$ to $\eta(x)$ where $\eta(x)$ satisfies

$$y(x) = -V'(x) + g(x)\xi(x) + \eta(x). \tag{53}$$

This is similar to the Langevin equation (1) but here $\xi(x)$ and $\eta(x)$ are not noise terms but smoothly varying functions of x. In (1) ξ and η are random variables while in (53) they are the particular statistical realizations of these random variables which give rise to the extremal path. Using (53) in (32) and (33) one arrives, after some algebraic manipulation, at the following differential equations in ξ and η:

$$\eta'(\eta - V' + g\xi) = (V'' - g'\xi)\eta \tag{54}$$

and

$$\tau^2\xi''(\eta - V' + g\xi)^2 = \xi - \tau^2\xi'[g\eta\xi' + (-V' + g\xi)(-V'' + g'\xi + g\xi')] - \frac{\sigma_2}{\sigma_1} g\eta. \tag{55}$$

In deriving these equations it was necessary to perform one differentiation so one boundary condition will be required in addition to (51) and (52). Substituting the particular value $\xi(x_0) = 0$ into (53) we see that the required extra boundary condition is just

$$\eta(x_0) = 0. \tag{56}$$

In terms of η and ξ the condition (52) at the endpoint x is

$$\xi + \tau(-V' + g\xi + \eta)\xi' = 0. \tag{57}$$

We solved (54) and (55) numerically in the range $(1, x)$ for several values of $x > 0$, using the form (42) for $V(x)$ and $g(x) = x$, and subject to the boundary conditions $\xi = 0$, $\eta = 0$ at $x = 1$ as well as the condition (57) at the endpoint x. The software routine used to integrate the differential equations was COLSYS [14] which must be supplied with an initial estimate for the solution. In practice it was necessary to guess values for ξ and η at x and then to vary these until (57) was satisfied. A graph of the action $S(x)$ for various τ, found by substituting the numerically derived solution into (31), is shown in figure 1.

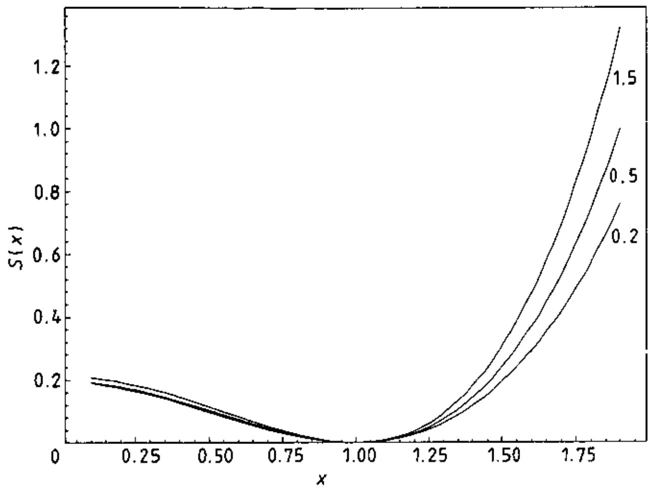

Figure 1. The action $S(x)$ plotted for the values of $\tau=0.2$, 0.5 and 1.5.

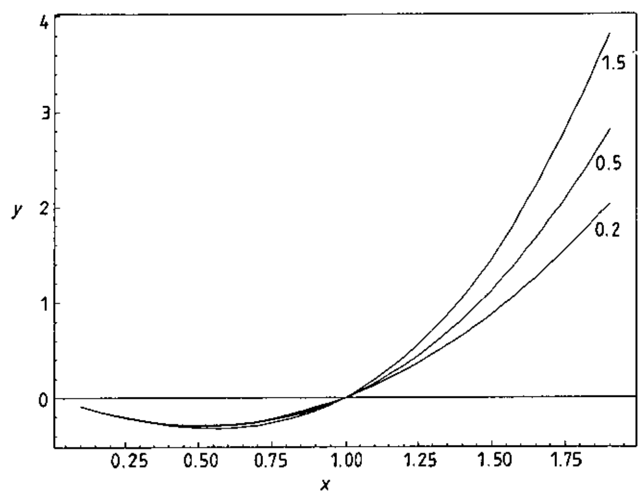

Figure 2. The value of y at the end point x calculated from (54) and (55) using the boundary condition (59) for $\tau=0.2$, 0.5 and 1.5.

Since both sides of (32) is a difference of squares it can be written in the form

$$(\xi-\tau y)(\xi+\tau y)=\frac{\sigma_2}{\sigma_1}(y-V'+g\xi)(y+V'-g\xi). \tag{58}$$

Substituting (52) into this implies that either

$$y-V'+g\xi=0 \tag{59}$$

or

$$y+V'-g\xi=0 \tag{60}$$

at the endpoint x. In all the cases considered it was found that the former of these, (59), was equivalent to (52). In the numerical work we used this simpler boundary condition instead of (57) and then checked that the solution obtained corresponded to a minimum with respect to ξ of the action $S(x, \xi)$. The numerical values of y which were found to obey (59) are plotted as a function of x in figure 2. Note that this is not a graph of the solution $y(x)$ for one particular endpoint but the value of y corresponding to and evaluated at different endpoints x of the interval $(1, x)$.

The numerical results for $S(x)$ give curves for the stationary probability distribution similar to those derived in [6] and [7] by other methods, though the range of values for the parameters D and R used by those authors differ from ours which precludes a direct comparison. In [6] the process described by (1) is treated as an approximate Markov process by means of an effective Fokker–Planck equation while in [7] a small τ expansion is made about the white noise limit. Our method tackles the essential non-Markovian nature of the coloured noise and makes no restriction on the value of τ. Furthermore, the method of steepest descents, used in this paper to leading order, can be extended in a systematic way [2] in order to obtain the prefactors multiplying the exponentials in the expressions for the stationary probability distribution and the mean escape time. In the case of additive coloured noise the prefactor for the escape rate has been calculated for small τ [13].

It has been shown [7, 15] that the laser system considered in this paper exhibits a first-order phase transition at certain values of D and τ. The phase transition only occurs at finite D and, since our results are derived in the $D \to 0$ limit, we are unable to explore this phenomenon. The model for the dye laser with $V(x)$ given by (42) and $g(x) = x$ can be derived, via a small intensity expansion, from the laser model considered in [16]. The latter model, unlike the one considered here, exhibits a first order phase transition for $\tau = 0$ as well as $\tau \neq 0$ and is therefore a more suitable starting point for the investigation of the phase transition.

6. Conclusion

We have shown how path integral methods, such as those developed in [9] to deal with additive coloured noise, can be used to study more complex systems such as the one described by (1). The method of steepest descents has been used in this case to obtain results in the small D limit where D is the diffusion constant. Advantages of the technique over others are that it is valid for general τ and that a systematic procedure for going to next order by considering fluctuations about the extremal path exists [2, 13].

To illustrate the use of path integral techniques, we have assumed that the diffusion constants D and R in the model are small and of the same order, but this is not always the case for the dye laser. In [17] it was found that simulations of the dye laser reproduced experimental behaviour when R was much smaller than D, so that in the dimensionless units used in this paper we would have, for example, $R = \mathrm{O}(10^{-6})$ and $D = \mathrm{O}(10^{-2})$. This suggests that a good approximation would be to set R equal to zero, in which case the model reduces to that described by the Langevin equation with a single multiplicative coloured noise term which is discussed in [4]. We are currently investigating these, and related, questions.

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References

[1] Doering C R, Brand H R and Ecke R E 1989 (eds) *J. Stat. Phys.* **54** 1111–1517
[2] Caroli B, Caroli C and Roulet B 1981 *J. Stat. Phys.* **26** 83
Weiss U 1982 *Phys. Rev.* A **25** 2444
[3] McKane A J 1989 *Phys. Rev.* A **40** 4050
[4] Bray A J, McKane A J and Newman T J 1990 *Phys. Rev.* A **41** 657
[5] Haken H 1984 *Laser Theory* (New York: Springer)
[6] Fox R F and Roy R 1987 *Phys. Rev.* A **35** 1838
[7] Aguado M and San Miguel M 1988 *Phys. Rev.* A **37** 450
[8] Stein D L, Doering C R, Palmer R G, van Hemmen J L and McLaughlin R M 1990 *J. Phys. A: Math. Gen.* **23** L203
[9] McKane A J, Luckock H C and Bray A J 1990 *Phys. Rev.* A **41** 644
[10] Gardiner C W 1985 *Handbook of Stochastic Methods* 2nd edn (Berlin: Springer)
[11] Graham R 1977 *Z. Phys.* B **26** 281
[12] van Kampen N G 1981 *J. Stat. Phys.* **24** 175
[13] Luckock H C and McKane A J 1990 *Phys. Rev.* A **42** 1982
[14] Ascher U, Christiansen J and Russell R D 1981 *ACM Trans. Math. Software* 7 209
[15] Lett P, Gage E C and Chyba T H 1987 *Phys. Rev.* A **35** 746
[16] Peacock-Lopez E, Javier de la Rubia F, West B J and Lindenberg K 1989 *Phys. Rev.* A **39** 4026
[17] Yu A W, Agrawal G P and Roy R 1989 *J. Stat. Phys.* **54** 1223